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The accelerating universe in brane-world cosmology

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Abstract

The standard Friedmann universe embedded in a five-dimensional bulk with constant curvature is examined as a brane-world with the extrinsic curvature derived directly from Codazzi's equation. It is shown that the accelerated expansion of the universe can be described as an extrinsic geometrical property, as an alternative to dark energy.

1. Introduction

The 1997 and subsequent observations of type Ia supernovae suggest that the universe is presently undergoing an accelerated expansion [1,2]. This phenomenon indicates the existence of an energy component characterized by a negative pressure, contributing with about 70% of the total energy density of the universe generally known as dark energy, the other 30% being essentially non-relativistic matter. In addition to the most natural candidate to explain such a component, the cosmological constant, many phenomenological models with a negative pressure, like scalar fields with dominant potential energy (quintessence) [3] or x-matter [4], have been proposed.

Although interesting, in general the dynamics of dark energy candidates has no justification but phenomenological ones. In particular, as the densities of the dark energy and of the other energy components decrease at different rates, the near coincidence in their values today can be explained only phenomenologically, either by a fine tuning of initial conditions or by a careful choice of "tracking" potentials [5]. Other proposals consider couplings between the dark components, such that the universe present an asymptotic regime with fixed dark matter to dark energy ratio [6]. In spite of any kinematical advantages of such approaches, it is indisputable that any form of dark energy should have its dynamics derived from an underlying theory.

The brane-world program primarily aimed to solve the hierarchy problem of fundamental interactions, may provide such a theory. It started with the observation that the commonly accepted hypothesis that gravitation becomes strong only at around 10^{19} GeV is a

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conjecture devoid of any experimental support, and that there is no theoretical or experimental evidence which prevents the existence of quantum gravity effects up to the TeV scale of energies [7].

Brane-world gravity differs from Einstein's gravity in some relevant aspects. TeV gravitons are allowed to propagate in the higher-dimensional bulk, but the electromagnetic, weak and strong interactions together with ordinary matter remain confined to the four-dimensional brane manifold. The geometrical scenario is that of a four-dimensional manifold which is embedded and evolves in a higher-dimensional bulk. The purpose of this note is to show how such a new and rich geometry, as compared with the pure Riemannian case, affects the behavior of the universe and in particular how the associated extrinsic geometry may be related to the accelerated expansion of the universe.

Several brane-world cosmologies in five dimensions have been proposed, some of them defined in an anti-de Sitter bulk (AdS_5), featuring boundary terms in the action, or regarding the brane as a single boundary of the bulk, with or without mirror symmetries as in the Randall–Sundrum (RS) formulations [8]. Other brane-world cosmological models explore the presence of additional fields in the bulk [9] and more recently a flat bulk is considered, using the fundamental energy scale as a free parameter [10].

The interaction between the bulk and the brane is not a trivial issue. Indeed, when a graviton or gravitational wave crosses the brane-world it is subjected to a deviation, in a process that is similar to the crossing of an electromagnetic wave by a charged surface. The difference is that for gravitons this is expressed in terms of the extrinsic curvature k_{ij} of the embedded geometry, representing the tangent components of the local variation of the normal unit vector (defined in the next section). The most commonly used such condition is the Israel–Lanczos condition [11]

$$k_{ij} = -\frac{1}{2}\alpha_5^2\left(T^m_{ij} - \frac{1}{3}T^m g_{ij}\right), \tag{1}$$

where α_5 is a constant proportional to the bulk gravitational constant and T^m_{ij} is the energy–momentum tensor of confined matter. The main difficulty in applying this condition is that it is not unique. Other forms of junction conditions exist, also relating the extrinsic curvature to the energy–momentum tensor [12], so that the application of a different condition may lead to different physical results. The use of (1) in RS models and other brane-world cosmologies usually require further fixes to make them compatible with the big bang nucleosynthesis [13].

To understand when and why we must apply a junction condition we should first look at the meaning of the extrinsic curvature. Since this represents a measure of the local deviation from the manifold and its local tangent space, it provides a more detailed description of the local shape of the geometry than the use of Riemannian geometry alone. That is, in the same manner that it is not always possible to distinguish between a plane, a cylinder or a cone by just using the intrinsic geometry, it is also not possible to distinguish the Friedmann–Robertson–Walker (FRW) model from a similar one, with the same metric but with different extrinsic properties, by use of Riemannian geometry alone. We want to find if this extrinsic curvature has anything to do with the accelerated expansion.

Since junction conditions like (1) is an *algebraic* statement on the behavior of the extrinsic curvature in terms of the energy–momentum tensor of the confined sources. The net result is to replace any eventual dynamics that the extrinsic curvature could have on the evolution of the universe by the confined matter. Therefore, in order to make explicit the role of the extrinsic curvature we will delay the application of (1) as much as possible.

2. The FRW universe as a brane-world

In what follows we will analyze the influence of the extrinsic curvature to the FRW universe, regarded as a brane-world embedded in a five-dimensional constant curvature bulk. We show that the accelerated expansion may be explained as an effect of the non-vanishing extrinsic curvature of the universe.

The FRW line element is written as

$$ds^2 = -dt^2 + a^2\left[dr^2 + f(r)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right], \tag{2}$$

where $f(r) = \sin r, r, \sinh r$ corresponding to $k = 1, 0, -1$, respectively. It is well known that this space–time can be embedded into a five-dimensional flat space [14], but here, to suit the different bulks used in the literature we extend this embedding to any constant curvature bulk, including the de Sitter dS_5, the anti-de Sitter AdS_5 and the flat M_5 manifolds. In all cases we fix the bulk metric signature to $(4,1)$, so that the bulk Riemann tensor can be generically written as[1]

$$ {}^5\mathcal{R}_{\mu\nu\rho\sigma} = -\frac{\Lambda_5}{6}(\mathcal{G}_{\mu\rho}\mathcal{G}_{\nu\sigma} - \mathcal{G}_{\mu\sigma}\mathcal{G}_{\nu\rho}), \tag{3} $$

where $\mathcal{G}_{\mu\nu}$ denotes the bulk metric and Λ_5 is a bulk cosmological constant which can be positive, negative or zero in the flat bulk case.

The embedding itself is determined by the components $\mathcal{Z}^\mu$ of a map $\mathcal{Z}: V_4 \to V_5$ such that

$$ \mathcal{Z}^\mu_{,i}\mathcal{Z}^\nu_{,j}\mathcal{G}_{\mu\nu} = g_{ij}, \qquad \mathcal{Z}^\mu_{,i}\eta^\nu\mathcal{G}_{\mu\nu} = 0, $$
$$ \eta^\mu\eta^\nu\mathcal{G}_{\mu\nu} = 1. \tag{4} $$

Here g_{ij} denote the four-dimensional metric in arbitrary coordinates and η^μ are the components of the unit vector orthogonal to the brane-world. Equivalently, the bulk metric components in the bulk vielbein $\{\mathcal{Z}^\mu_{,i}, \eta^\mu\}$ can be summarized as

$$ \mathcal{G}_{\mu\nu} = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{55} \end{pmatrix}, \qquad g_{55} = 1. \tag{5} $$

The Gauss and Codazzi equations for the embedding in five dimensions are respectively [15]

$$ R_{ijk\ell} = (k_{ik}k_{jl} - k_{il}k_{kj}) + {}^5\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{Z}^\mu_{,i}\mathcal{Z}^\nu_{,j}\mathcal{Z}^\rho_{,k}\mathcal{Z}^\sigma_{,\ell}, \tag{6} $$
$$ k_{ij;k} - k_{ik;j} = 0, \tag{7} $$

where the components of the extrinsic curvature are given by $k_{ij} = -\eta^\mu_{,i}\mathcal{Z}^\nu_{,j}\mathcal{G}_{\mu\nu}$. Now, assuming that the embedding is regular we may derive from (4) the inverse expression

$$ g^{ij}\mathcal{Z}^\mu_{,i}\mathcal{Z}^\nu_{,j} = \mathcal{G}^{\mu\nu} - \eta^\mu\eta^\nu. $$

Thus, we obtain from the contractions of (6) with g^{ij}

$$ {}^5\mathcal{R} = R - (K^2 - h^2) + 2\,{}^5\mathcal{R}_{\mu\nu}\eta^\mu\eta^\nu, \tag{8} $$

where $h = g^{ij}k_{ij}$ denotes the mean curvature of the brane-world and $K^2 = k^{ij}k_{ij}$. Consequently, the Einstein–Hilbert Lagrangian of the bulk decomposes as [16]:

$$ {}^5\mathcal{R}\sqrt{-\mathcal{G}} = R\sqrt{-g} - (K^2 - h^2)\sqrt{-g} + 2\,{}^5\mathcal{R}_{\mu\nu}\eta^\mu\eta^\nu\sqrt{-g}. $$

The corresponding Euler–Lagrange equations with respect to g_{ij} gives the brane equations of motion (after adding the source term $T^m{}_{ij}$)

$$ R_{ij} - \frac{1}{2}Rg_{ij} = Q_{ij} + \left({}^5\mathcal{R}_{\mu\nu} - \frac{1}{2}\,{}^5\mathcal{R}\mathcal{G}_{\mu\nu}\right)\mathcal{Z}^\mu_{,i}\mathcal{Z}^\nu_{,j} + 8\pi G T^m{}_{ij}, $$

where we have denoted

$$ Q_{ij} = g^{mn}k_{im}k_{jn} - hk_{ij} - \frac{1}{2}(K^2 - h^2)g_{ij}. \tag{9} $$

This quantity is defined by the extrinsic curvature and it does not exist in Einstein's equations as defined in pure Riemannian geometry. Using (3) we obtain

$$ \left({}^5\mathcal{R}_{\mu\nu} - \frac{1}{2}\,{}^5\mathcal{R}\mathcal{G}_{\mu\nu}\right)\mathcal{Z}^\mu_{,i}\mathcal{Z}^\nu_{,j} = -\Lambda_5 g_{ij}. $$

Therefore, the equations of motion for a brane-world compatible with the embedding in a five-dimensional constant curvature bulk are

$$ R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda_5 g_{ij} = 8\pi G T^m{}_{ij} + Q_{ij}. \tag{10} $$

One important conclusion is that

$$ Q^{\mu\nu}{}_{;\nu} = 0, \tag{11} $$

is a consequence derived directly from (9). Together with the conservation of $T^m{}_{ij}$, it follows that whatever is the influence of this new term on the evolution of the brane, it turns out to be energetically uncoupled from the other components of the universe.

3. Dark energy as geometry

In order to explicitly calculate Q_{ij}, we explore an interesting feature that in five dimensions Codazzi's equation can be solved separately. From its definition, it follows that for diagonal metrics k_{ij} is also diagonal. In this case the solution of (7) becomes particularly simple. In fact, denoting the spatial indices in the brane by the letters $a, b, c, d = 1, \ldots, 3$, we find that (7) can be separated as [16,17]

$$ k_{aa,c} - k_{ad}\Gamma^d_{ac} = k_{ac,a} - k_{cd}\Gamma^d_{aa}, $$
$$ k_{aa,4} - k_{aa}\frac{\dot a}{a} = -a\dot a\left(\delta^1_a\delta^1_b + f^2\delta^2_a\delta^2_b + f^2\sin^2\theta\,\delta^3_a\delta^3_b\right)k_{44}. $$

[1] Greek indices go from 1 to 5 and refer to the bulk, small case Latin indices go from 1 to 4 and refer to the brane. ${}^5\mathcal{R}$ denotes diverse Riemann curvatures of the bulk.

The first equation gives $k_{11,c} = 0$, so that k_{11} is a function of t only, denoted by $b(t)$ and called the "radial bending" of the brane-world. From the second equation we obtain

$$k_{44} = -\frac{1}{\dot{a}}\frac{d}{dt}\left(\frac{b}{a}\right).$$

Repeating the same arguments for k_{22} and k_{33} we obtain the general solution

$$k_{ab} = \frac{b}{a^2}g_{ab} \quad \text{and} \quad k_{44} = -\frac{1}{\dot{a}}\frac{d}{dt}\left(\frac{b}{a}\right). \tag{12}$$

Denoting $B = \dot{b}/b$ and, as usual the Hubble parameter $H = \dot{a}/a$, we obtain from (9)

$$Q_{ab} = \frac{b^2}{a^4}\left(2\frac{B}{H} - 1\right)g_{ab}, \qquad Q_{44} = -\frac{3b^2}{a^4}. \tag{13}$$

Replacing this expression in (10) and after eliminating $\ddot{a}$, we obtain the Friedmann's equation as modified by the presence of the extrinsic curvature and the bulk constant curvature Λ_5

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda_5}{3}a^2 + \frac{b^2}{a^2}, \tag{14}$$

where the radial bending $k_{11} = b(t)$ remains arbitrary. If we assume $b(t) = 0$, then we recover the usual Friedmann's equation with a cosmological term.

Referring to our previous discussion on the use of junction conditions, we may now check the compatibility of (14) with the use of the Israel–Lanczos condition (1) applied to our solution (12). Calculating $k_{11} = b$ from (1) we find that $b(t) = -\frac{1}{6}\alpha_5^2\rho a^2$. Replacing this in (14) we obtain Friedmann's equation with the square of the energy density

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda_5}{3}a^2 + \frac{\alpha_5^4}{36}\rho^2 a^2. \tag{15}$$

Clearly showing that contrarily to a common statement, Friedmann's equation in brane-worlds does not necessarily imply the presence of a ρ^2 term, but only after we apply the Israel–Lanczos junction condition.

The above derivation of the ρ^2 term does not depend on any assumption other than the embedding of the FRW cosmology and the use of (1). It is not a property of RS and other models, where (1) would necessarily apply. That is, when there are boundaries in the bulk and/or the use of Z_2 symmetries is required among other possible assumptions. If we leave out these restrictions, then we obtain a more general scenario represented by (14), where the modifying term in Friedmann's equations is just a component of the extrinsic curvature.

To proceed with this geometrical interpretation let us make an analogy between k_{ij} and a "fluid" with energy–momentum tensor

$$\tau_{ij} \equiv -\frac{Q_{ij}}{8\pi G}. \tag{16}$$

As a consequence of (11), it follows that this "fluid" does not exchange energy with the ordinary confined matter. Denoting by p_b its pressure and by ρ_b the corresponding energy density, then we can represent τ_{ij} as

$$\tau_{ij} = (p_b + \rho_b)U_i U_j + p_b g_{ij}, \qquad U_i = \delta_i^4, \tag{17}$$

to which we add a state-like equation $p_b = (\gamma_b(t) - 1)\rho_b$, where $\gamma_b(t)$ remains an undetermined function of time. Comparing τ_{ab} and τ_{44} to (13), we can express p_b and ρ_b in terms of b as

$$\rho_b = \frac{3}{8\pi G}\frac{b^2}{a^4},$$
$$p_b = -\frac{1}{8\pi G}\frac{b^2}{a^4}\left(2\frac{B}{H} - 1\right), \tag{18}$$

so that

$$Q = g^{ij}Q_{ij} = 3p_b - \rho_b = g^{ij}Q_{ij} = \frac{6b^2}{a^4}\frac{B}{H}.$$

Replacing p_b in the trace expression and using the equation of state we obtain the equation for $b(t)$:

$$\frac{\dot{b}}{b} = \frac{1}{2}\big(4 - 3\gamma_b(t)\big)\frac{\dot{a}}{a}, \tag{19}$$

which is analogous to the x-matter, one of the phenomenological candidates for dark energy [4]. This suggests the brane extrinsic curvature as a possible fundamental explanation for such models.

In order to further explore this analogy consider a simple example where γ_b is taken as a constant. In this case, the above equation yields a very simple solution

$$b(t) = b_0 a(t)^{\frac{1}{2}(4-3\gamma_b)}, \tag{20}$$

where b_0 is an integration constant. With this solution, the "bending" energy becomes

$$\rho_b = \frac{3b_0}{8\pi G}a^{-3\gamma_b}. \tag{21}$$

For a simple estimate, consider a vanishing Λ_5 corresponding to a flat bulk and a spatially flat ($k=0$) FRW brane-world, composed mainly by dark matter and the bending contribution in place of the dark energy. In this case, denoting $\Omega_m \equiv \frac{8\pi G\rho_m}{3H^2}$ (for the dark matter) and $\Omega_b \equiv \frac{8\pi G\rho_b}{3H^2}$ (for the geometric contribution), the partial deceleration parameter is

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = (3\gamma_b - 2)\frac{\Omega_b}{2} + \frac{\Omega_m}{2}. \tag{22}$$

Thus, for $\Omega_m \sim 0.3$ and $\Omega_b \sim 0.7$, as suggested by recent observations, a present time universe driven by the extrinsic curvature occurs whenever $\gamma_b < 0.52$ as in the x-matter case [4]. Although this result is not new, the possible geometrical interpretation for the x-matter allows us to consider observational tests as indirect measurements of the extrinsic curvature (and its evolution) for constant values of γ_b. The cases with γ_b not constant may require an improved approach, perhaps with the use of reconstruction techniques for the equation of state of this geometrical dark energy [18]. This is an inversion with respect to the usual approaches in the sense that observational data may be used to measure the evolution of geometry, and such results may facilitate the elaboration of models explaining the bending of the brane-world, or, in other words, the evolution of the dark energy component.

4. Summary

Using only the geometrical properties of brane-worlds and without appealing to a junction condition, we have shown, for the simple case of FRW metric embedded in a five-dimensional bulk with constant curvature, that the unknown dark energy component that drives the acceleration of the universe can be associated to its extrinsic curvature. This geometrical interpretation may be advantageous over other dark energy proposals because we may apply some yet unexplored geometrical properties, such as a dynamical equation for the extrinsic curvature or a perturbative analysis of the FRW model along the extra dimensions (just to cite two examples) in the search for viable models.

One of our initial hopes was to obtain a solution of the coincidence problem through such interpretation. This has not been exhausted but the derivation of Eq. (11), tells that the extrinsic geometry is in fact decoupled from the confined matter, independently of any dark energy interpretation which we may give to this geometry.

However, it should be noted from Gauss' equation (6), or more particularly from the dynamical equations (10), that the extrinsic curvature does not evolve independently of the source terms in the brane. Thus, even without a direct coupling, a new solution to the coincidence problem written in geometrical terms can in principle be attempted.

On the other hand the participation of the extrinsic geometry on the brane dynamics as can be seen from (10) means that it responds in a non-trivial way to the presence of matter. Admittedly, an additional dynamical equation for k_{ij} is on demand.

The extra term in Friedmann's equation was shown to behave as x-matter, the phenomenological models proposed to fit data of the universe acceleration, but lacking of a more fundamental justification. Again, the geometrical interpretation has the advantage that whatever are the constraints on x-matter imposed by the data, it will be absorbed as a geometrical condition on the brane.

The present geometrical interpretation may help in the construction of more specific geometrical models of x-matter or, at least, to bring some new insights into the dark energy paradigm, which would otherwise be difficult to attain.

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